Student Number : ____________________

YEAR 12
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2021

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper (pages 21-24)
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 8)

- Attempt Questions 1 – 10
- Answer on the multiple choice sheet provided on page 19 (you may detach this sheet)
- Allow about 15 minutes for this section

Section II – 60 marks (pages 9 – 18)

- Attempt Questions 11 – 14
- Answer each question in a separate answer booklet
- Allow about 1 hours and 45 minutes for this section

Section 1 – 10 Marks (1 mark each)

- Attempt Questions 1-10
- Answer on the Multiple Choice Sheet provided on Page 19
  (You may detach this sheet)
- Allow about 15 minutes for this section

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1. Which of the following is NOT a factor of the polynomial
   $P(x) = x^3 - x^2 - 8x + 12$?

(A) $(x+3)$

(B) $(x+3)^2$

(C) $(x-2)$

(D) $(x-2)^2$

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2. What is the primitive of $\dfrac{\ln 2}{\sqrt{\pi - x^2}}$?

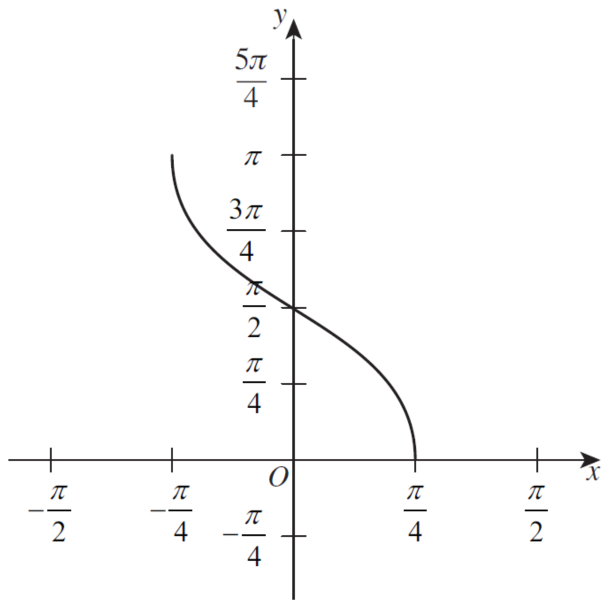
(A) $(\ln 2)(x)\left(\sin^{-1}\left(\dfrac{x}{\sqrt{\pi}}\right)\right) + C$

(B) $(\ln 2)\left(\sin^{-1}\left(\dfrac{x}{\sqrt{\pi}}\right)\right) + C$

(C) $\left(\dfrac{\ln 2}{\sqrt{\pi}}\right)\left(\sin^{-1}\left(\dfrac{x}{\sqrt{\pi}}\right)\right) + C$

(D) $(\ln 2)\left(\sin^{-1}\left(\dfrac{x}{\pi}\right)\right) + C$

3. The graph of an inverse trigonometric function is shown.



What is the equation of this function?

(A) $\dfrac{4}{\pi}\cos^{-1} x$

(B) $\dfrac{\pi}{4}\cos^{-1} x$

(C) $\cos^{-1}\left(\dfrac{4x}{\pi}\right)$

(D) $\cos^{-1}\left(\dfrac{\pi x}{4}\right)$

4. A bag contains 2 red, 5 blue, 6 white, 11 green and 14 yellow marbles. What is the minimum number of marbles that need to be chosen randomly from the bag to ensure that 6 marbles of the same colour have been chosen?

(A) 16

(B) 17

(C) 23

(D) 34

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5. The function $f(x) = x^2 - 4x + 7$ has an inverse function $f^{-1}(x)$ in the domain $x \geq 2$.

What is the value of $f^{-1}\left(f(m)\right)$,
where m is a real number NOT in the domain $x \geq 2$?

(A) m

(B) $2 - m$

(C) $2 + m$

(D) $4 - m$

6. The vectors $4\underset{\sim}{i} - \underset{\sim}{j}$ and $3\underset{\sim}{i} + m\underset{\sim}{j}$ are perpendicular.
What is the value of m?

(A) -12

(B) -1

(C) 3

(D) 12

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7. A piece of hot metal is placed in a room with a surrounding air temperature of $15°C$ and is allowed to cool.
It loses heat according to Newton's Law of Cooling:

$$\frac{dT}{dt} = -k\left(T - A\right) \text{ where}$$

- T is the temperature of the metal in $°C$, at time t minutes
- A is the surrounding air temperature
- k is a positive constant

After 5 minutes the temperature of the metal is $75°C$
and after a FURTHER 3 minutes the temperature of the metal is $45°C$.

What is the value of k in the above equation?

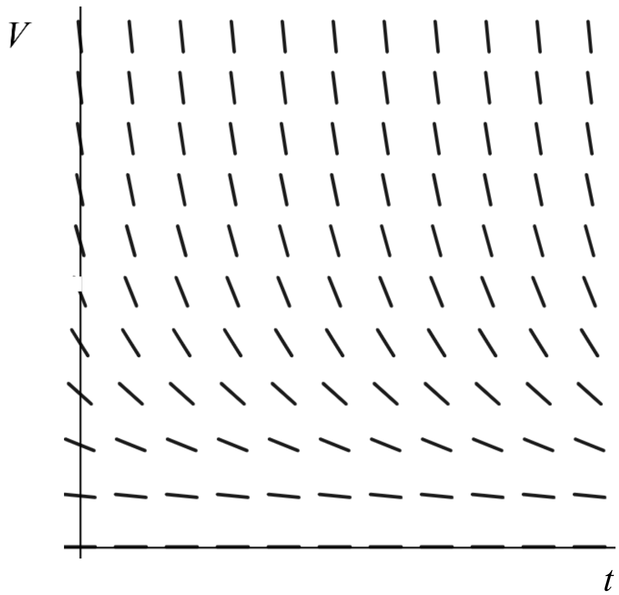
(A) $3\ln\left(0.5\right)$

(B) $3\ln\left(2\right)$

(C) $\dfrac{\ln\left(0.5\right)}{3}$

(D) $\dfrac{\ln\left(2\right)}{3}$

8. A direction field for the volume of water, V megalitres, in a reservoir t years after 2021 is shown below.



According to this model, for $k > 0$, what is the value of $\dfrac{dV}{dt}$?

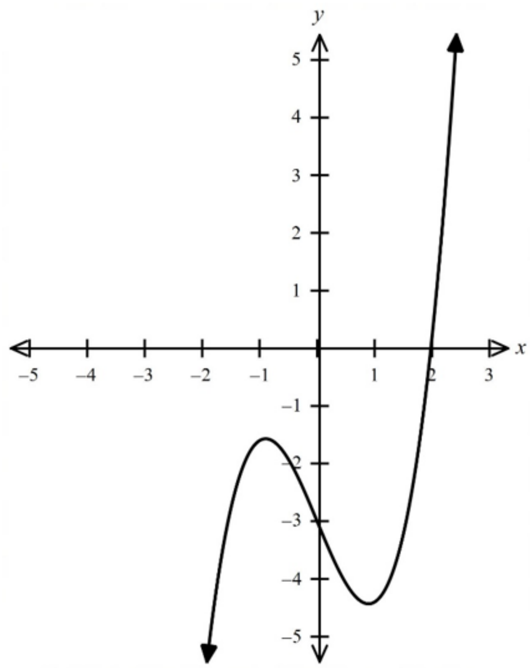
(A) $-kV^2$

(B) $-\dfrac{k}{V}$

(C) $\dfrac{k}{V}$

(D) kV^2

9. Consider the graph of $y = f(x)$ shown below:



Which of the following would have two MORE roots than $f(x)$?

(A) $y = -2 \times f(x)$

(B) $y = f(x) + 3$

(C) $y = f^{-1}(x)$

(D) $y = f(x + 3)$

10. It is known that $\sin x = \frac{1}{3}$, where $\frac{\pi}{2} < x < \pi$

What is the value of $\cos(2x)$?

(A) $-\frac{7}{9}$

(B) $\frac{7}{9}$

(C) $\frac{2}{3}$

(D) $-\frac{4\sqrt{2}}{3}$

END OF SECTION 1

SECTION 2 – 60 Marks

- Attempt Questions 11-14
- Allow about 1h 45min for this section
- Answer each question in a separate writing booklet (extra writing booklets are available upon request from the examination supervisor)
- Show all relevant mathematical reasoning and/or calculations

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Marks

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) (i) Write $\cos(7x)\sin(5x)$ as a sum/difference of trigonometric ratios 1

(ii) Hence, solve $2\cos(7x)\sin(5x)=\sin(12x)-\dfrac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$ 3

(b) Prove by mathematical induction that $2^{2n}+6n-1$ is divisible by 3 for all integers $n \ge 1$ 3

(c) Consider the polynomial $P(x)=x^3-2px+q$ where p and q are constants and $p \ne 0$.

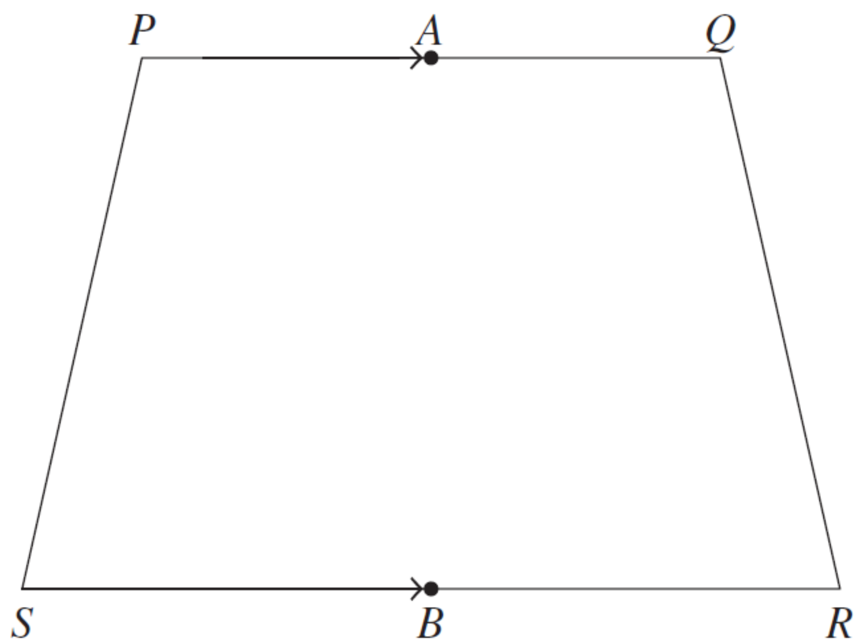
It is given that α, β and $\alpha+\beta$ are roots of the equation $P(x)=0$.

(i) Prove that $\alpha=-\beta$ 1

(ii) Hence, or otherwise, find all the roots of $P(x)=0$ in terms of p. 3

Question 11 continues on page 10

(d) $PQRS$ is a trapezium with A and B being the midpoints of the parallel sides PQ and RS respectively. Marks



Let $\overrightarrow{PA} = \underset{\sim}{a}$ and $\overrightarrow{SB} = \underset{\sim}{b}$.

(i) Express $\overrightarrow{QR}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\overrightarrow{AB}$ 2

(ii) Hence, or otherwise, show that $\overrightarrow{AB} = \dfrac{1}{2}\left(\overrightarrow{PS} + \overrightarrow{QR}\right)$ 2

END OF QUESTION 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Write the expression $\sqrt{3}\sin x + \cos x$ in the form $R\sin(x+\alpha)$, **2**

where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

(ii) Write down the three missing numbers that would appear in the cells of the table below, where $g(x) = \sqrt{3}\sin x + \cos x = R\sin(x+\alpha)$ **1**
and the numbers are given correct to 2 decimal places

x	$-\dfrac{\pi}{6}$	0	$\dfrac{2\pi}{6}$	$\dfrac{5\pi}{6}$	$\dfrac{8\pi}{6}$	$\dfrac{11\pi}{6}$	$\dfrac{14\pi}{6}$	$\dfrac{17\pi}{6}$	$\dfrac{20\pi}{6}$
$\ln x$			0.05	0.96	1.43	1.75			2.35
$g(x)$	0	1	2	0	-2	0	2	0	-2

(iii) Sketch the graphs of $y = \ln x$ and $y = g(x)$ on the same number plane, showing clearly any point(s) of intersection. **3**
(You do not need to give the coordinates of any point(s) of intersection, just show their location on your sketch)

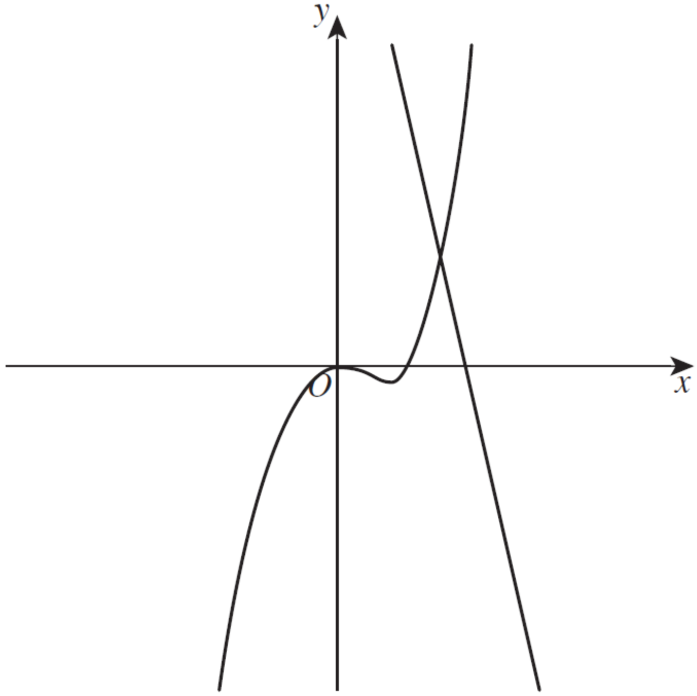
(iv) Hence, determine the number of solutions to the equation **1**
$\sqrt{3}\sin x + \cos x = \ln x$ in the domain $(0, \infty)$

(b) Use the identity $(1+x)^m(1+x)^n = (1+x)^{m+n}$ to show that **3**

$$\binom{m+n}{4} = \binom{n}{4} + \binom{n}{3}\binom{m}{1} + \binom{n}{2}\binom{m}{2} + \binom{n}{1}\binom{m}{3} + \binom{m}{4}$$

Question 12 continues on page 12

(c) The graphs of $g(x) = mx + b$ and $f(x) = 3x^3 - 2x^2$ are shown below



The solution to the equation $mx + b < 3|x|^3 - 2|x|^2$ is $x \in (-\infty, -2) \cup (1, \infty)$.

(i) Sketch the graph of $y = 3|x|^3 - 2|x|^2$ 1

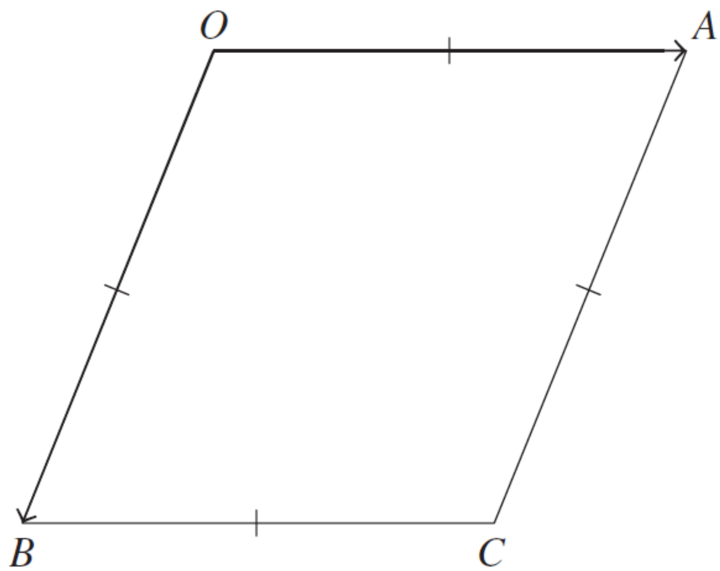
(ii) Hence, find the values of m and b. 3

(d) In how many different ways can 6 runners finish 1st, 2nd and 3rd in a race? 1
(assume no equal placings)

END OF QUESTION 12

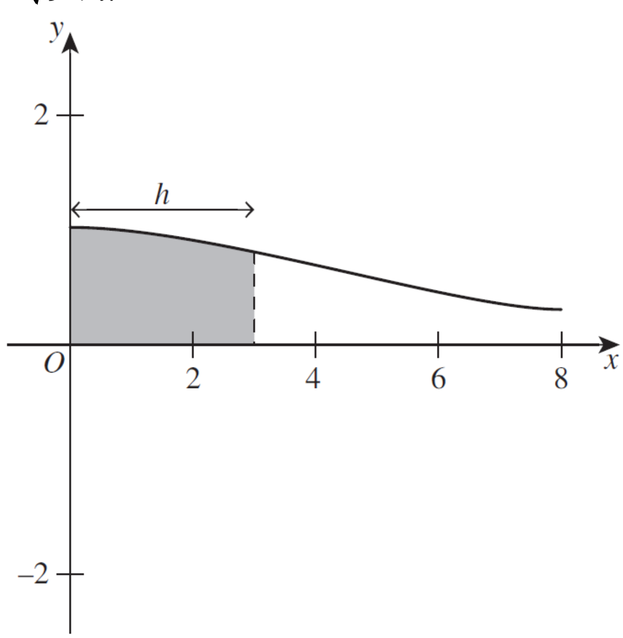
Question 13 (15 marks) Use a SEPARATE writing booklet — Marks

(a) $OACB$ is a rhombus with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ — 2



Using vector methods, prove that the diagonal OC bisects angle AOB.

(b) The graph of $y = \dfrac{3}{\sqrt{9+x^2}}$ for $x > 0$ is shown below.



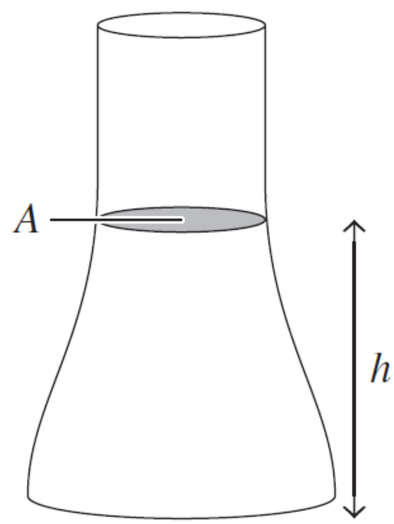
Question 13(b) continues on page 14

(i) The area bounded by $y = \dfrac{3}{\sqrt{9+x^2}}$, 2

the $x-$axis and the lines $x = 0$ and $x = h$ is rotated about the $x-$axis to create a solid of revolution.

Show that the volume of the solid is given by $V = 3\pi \tan^{-1}\left(\dfrac{h}{3}\right)$ cubic units.

(ii) The solid of revolution of the graph has the shape of a vase lying down on its side. 2
The diagram shows a vase of the same shape standing upright.



As water is being poured into the vase, the height h of the water in the vase is increasing at a rate of 3 cm/s.

Find the exact rate at which the volume of water in the vase, V, is increasing when $h = 6$.

(iii) The surface of the water forms a circular shape with area A.
By using the original graph, or otherwise, find the exact rate at which the area, A, is decreasing when $h = 6$. 2

Question 13 continues on page 15

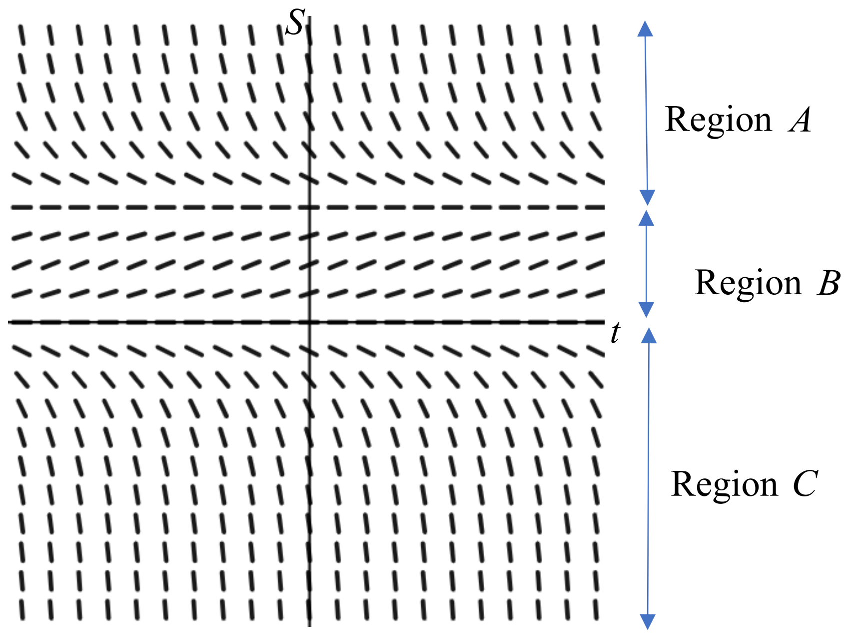
(c) The spread of flu through a student population is modelled by the equation

$S = \dfrac{2000}{1+199e^{-0.4t}}$ where S is the total number of students infected after t days

(i) Show that the given equation for S satisfies the differential equation 3

$\dfrac{dS}{dt} = \dfrac{S}{5}\left(2 - \dfrac{S}{1000}\right)$

(ii) The slope field of $\dfrac{dS}{dt} = \dfrac{S}{5}\left(2 - \dfrac{S}{1000}\right)$ is shown. 2



There are 3 regions labelled A, B and C in which a solution curve can be found.

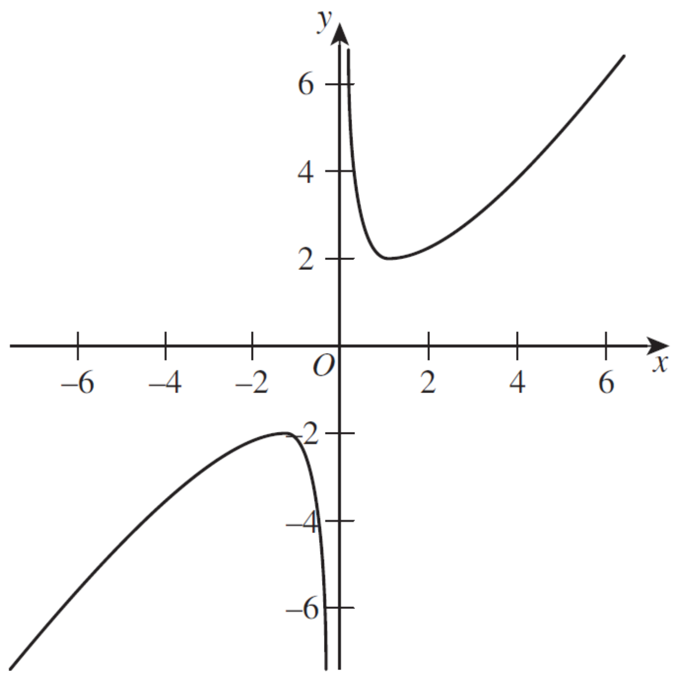
In which one of these three regions can the solution curve exist?
Justify your answers with reference to constant solutions and the initial value of S.

(iii) According to this model, after how many days does the rate of increase reach a maximum? Give your answer to the nearest number of days. 2

END OF QUESTION 13

Question 14 (15 marks) Use a SEPARATE writing booklet Marks

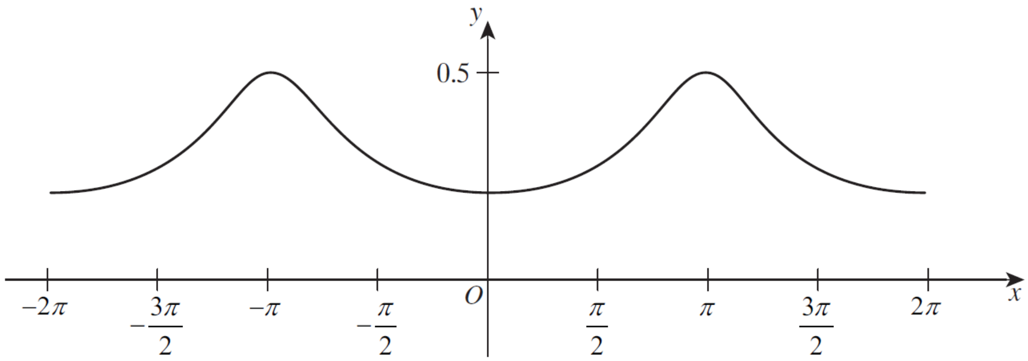
(a) The graph of the function $f(x) = \dfrac{x^2+1}{x}$ is shown 3



Sketch the graph of $y = \dfrac{1}{\sqrt{f(x)}}$ showing all important features including turning point(s), intercept(s) and asymptote(s).

Question 14 continues on page 17

(b) The graph of the function $y = \dfrac{1}{5+3\cos x}$ for $-2\pi \le x \le 2\pi$ is shown below:



(i) By using the substitution $t = \tan\left(\dfrac{x}{2}\right)$ and $t-$formula(s), show that **3**

$$\int \frac{1}{5+3\cos x}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\left(\frac{x}{2}\right)\right) + C$$ where C is a constant.

(ii) Hence, show that the area bounded by the curve $y = \dfrac{1}{5+3\cos x}$, **2**

the lines $x = 0$, $x = \pi$ and the $x-$axis is $\dfrac{\pi}{4}$

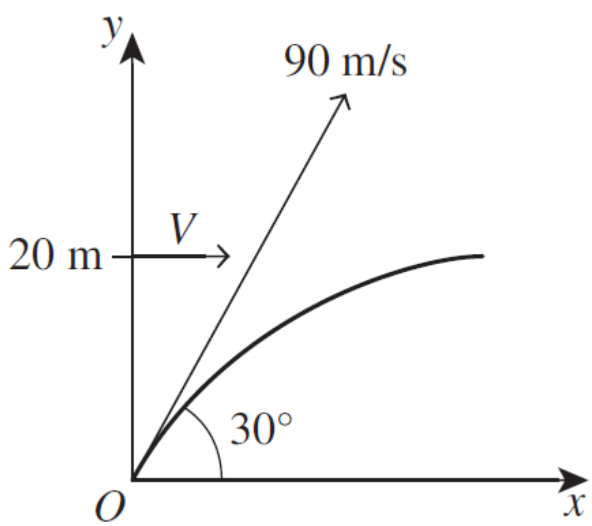
You may use the substitution $\tan\dfrac{\pi}{2} = \infty$ if necessary.

Question 14 continues on page 18

(c) (i) Initially a golf ball was hit from the ground at $90\ ms^{-1}$ at an angle of $30°$ to the horizontal and $g = 10\ ms^{-2}$. 2

Show that the position vector of the golf ball after t seconds is given by
$\underset{\sim}{s}(t) = \left(45\sqrt{3}t\right)\underset{\sim}{i} + \left(45t - 5t^2\right)\underset{\sim}{j}$

(ii) Five seconds after the golf ball was hit, a small stone was fired with velocity V horizontally from a point $20\ m$ above the ground. 3



By finding the position vector of the stone, or otherwise, show that the two objects collide after the golf ball has travelled for 21 seconds.

(iii) What is the stone's speed at collision?
Give your answer to 3 significant figures. 2

END OF EXAMINATION

1/ $P(2) = (2)^3 - (2)^2 - 8(2) + 12 = 0$

$\therefore$ $(x-2)$ is a factor

$P(-3) = (-3)^3 - (-3)^2 - 8(-3) + 12 = 0$

$\therefore$ $(x+3)$ is a factor

Product of roots is -12

$2^2 \times (-3) = -12$

$\therefore$ $(x-2)^2$ is a factor

so $(x+3)^2$ is not a factor

$\boxed{B}$

Note: Multiple choice logic

If $x+3$ was not a factor then $(x+3)^2$ wouldn't be. Similarly for $x-2$ & $(x-2)^2$. This means $(x+3)$ & $(x-2)$ are factors ... we can only eliminate one of the four options so it must be $(x-2)^2$ or $(x+3)^2$.

$2^2 \times (-3) = -12$

$2 \times (-3)^2 = 18$ $\therefore$ $(x+3)^2$ is NOT

$\therefore$ $\boxed{B}$

2/ $\displaystyle\int \frac{\ln 2}{\sqrt{\pi - x^2}}\,dx$

$$= \ln 2 \int \frac{1}{\sqrt{(\sqrt{\pi})^2 - (x)^2}}\,dx$$

Using $\displaystyle\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c$ we get

$$I = \ln 2 \,.\, \sin^{-1}\frac{x}{\sqrt{\pi}} + c$$

$\therefore$ $\boxed{B}$

or differentiate each option

eg A) $\displaystyle \ln 2 \times \left[x \times \frac{\frac{1}{\pi}}{\sqrt{1 - \left(\frac{x}{\pi}\right)^2}} + \sin^{-1}\frac{x}{\pi} \cdot 1 \right]$

$$= \ln 2 \,.\, x \, \frac{\frac{1}{\pi}}{\frac{\sqrt{\pi^2 - x^2}}{\pi}} + \sin^{-1}\frac{x}{\pi}$$

$$= \frac{\ln 2}{\sqrt{\pi^2 - x^2}} + \sin^{-1}\frac{x}{\pi} \quad \times$$

etc

3. Normally for $y=\cos^{-1}x$ $\quad -1\leq x\leq 1$, $\quad 0\leq y\leq \pi$

The range hasn't changed but now $-\frac{\pi}{4}\leq x\leq \frac{\pi}{4}$

So $\quad -1\leq ?\leq 1$

$$-\frac{\pi}{4}\leq x\leq \frac{\pi}{4}$$

$$-\pi\leq 4x\leq \pi$$

$$-1\leq \frac{4}{\pi}x\leq 1$$

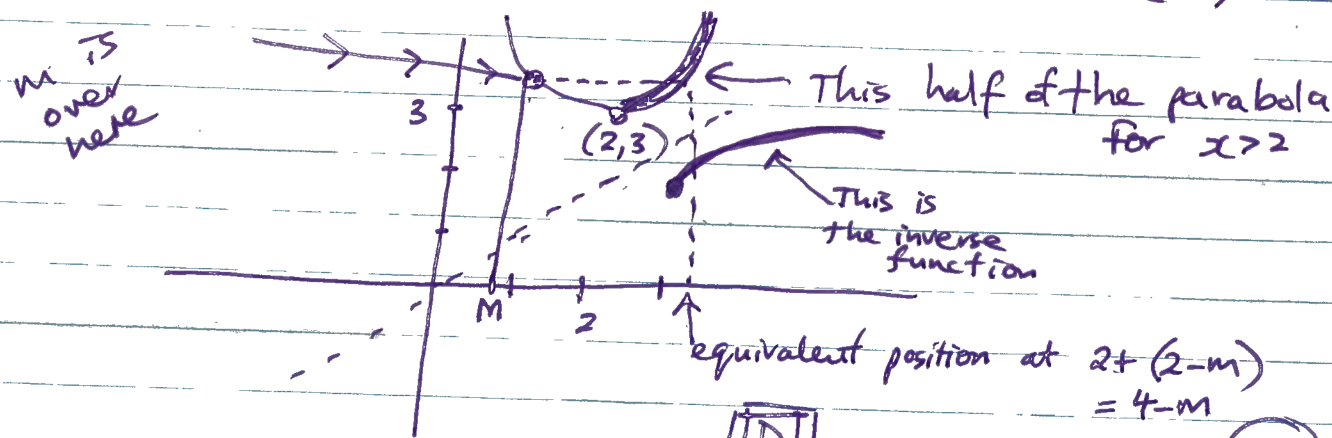
$\therefore \quad y=\cos^{-1}\left(\frac{4x}{\pi}\right)$ $\quad \boxed{C}$

4. You could choose

2 Red + 5 blue + 5 white + 5 green + 5 yellow and then the next marble will make 6 of one colour.

$\therefore \quad 23$ $\quad \boxed{C}$

5. $x^2-4x+7 = x^2-4x+2^2-2^2+7 = (x-2)^2+3$



$\boxed{D}$

6/ $(4)(3) + (-1)(m) = 0$ (dot product is zero for $\perp$ vectors)

$m = 12$

$\boxed{D}$

7/ $\frac{dT}{dt} = -k(T-15)$ from 1st sentence & second dot point

$\frac{dT}{T-15} = -k\,dt$

$\ln(T-15) = -kt + c$ (note: $\ln|T-15| = \ln(T-15)$ since $T>15$)

$\log_e(T-15) = -kt + c$

$e^{-kt+c} = T-15$

$me^{-kt} = T-15$ (let $m = e^c$)

$T = me^{-kt} + 15$

when $t = 5$ $T = 75$ $75 = me^{-5k} + 15$

when $t = 8$ $T = 45$ $45 = me^{-8k} + 15$

$60 = me^{-5k}$ .... (1)

$30 = me^{-8k}$ .... (2)

(1) $\div$ (2) $2 = \frac{e^{-5k}}{e^{-8k}}$

$2 = e^{3k}$

$\ln 2 = \ln e^{3k}$

$3k = \ln 2$

$k = \frac{\ln 2}{3}$

$\boxed{D}$

8/ $\frac{dv}{dt}$ is the gradient

From the direction field the gradient is always negative eg when $V=10$

Eliminate $\rightarrow$ C & D are positive

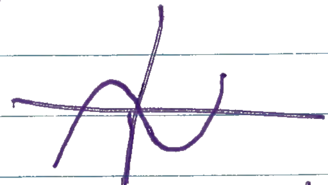
Now as V increases $\frac{dv}{dt}$ is more negative (greater in size)

The opposite would happen for B

$\therefore$ $\boxed{A}$

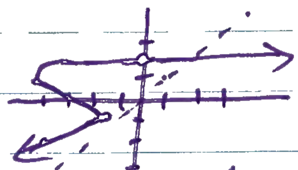
9/ A) Reflect in x-axis & make all points twice as far from x-axis
$\rightarrow$ still only 1 root

B) Translate curve up 3 units then



Yes 2 more roots ✓

c) Reflect in line $y=x$



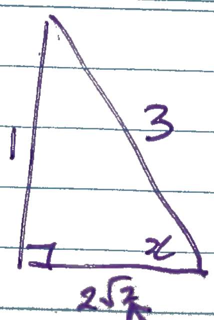
Coords "swap" $\therefore$ only root is at (-3,0

D) Translate 3 units left .. still 1 root

$\therefore$ $\boxed{B}$

5

10/



By Pythagoras

$\sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$

Now x is obtuse, so $\cos x = -\frac{2\sqrt{2}}{3}$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \left(\frac{-2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2$$
$$= \frac{8-1}{9}$$
$$= \underline{7}$$

$\boxed{B}$

11) a) i) $\cos 7x \sin 5x = \frac{1}{2}[\sin(7x+5x) - \sin(7x-5x)]$
$= \frac{1}{2}[\sin 12x - \sin 2x]$

ii) $2\cos 7x \sin 5x = \sin 12x - \frac{\sqrt{3}}{2}$

$2 \times \frac{1}{2}[\sin 12x - \sin 2x] = \sin 12x - \frac{\sqrt{3}}{2}$

$\sin 12x - \sin 2x = \sin 12x - \frac{\sqrt{3}}{2}$

$\sin 2x = \frac{\sqrt{3}}{2}$

$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

b) Show true for $n=1$

$2^{2(1)} + 6(1) - 1 = 4 + 6 - 1$
$= 9$
$= 3 \times 3$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie

$2^{2k} + 6k - 1 = 3M$, M is an integer

rearranging $2^{k} = 3M - 6k + 1$ *

Prove true for $n=k+1$ if true for $n=k$

7

ie RTP $2^{2(k+1)} + 6(k+1) - 1 = 3Q$ Q is an integer

$LHS = 2^{2k+2} + 6k + 6 - 1$
$= 2^{2k} \times 2^2 + 6k + 5$
$= (3M - 6k + 1) \times 4 + 6k + 5$ from $*$

$= 12M - 24k + 4 + 6k + 5$
$= 12M - 18k + 9$
$= 3(4M - 6k + 3)$
$= 3Q$ where $Q = 4M - 6k + 3$
and Q is an integer since
4, M, 6, k, 3 are integers

$\therefore$ $2^{2(k+1)} + 6(k+1) - 1$ is divisible by 3
if $2^k + 6k - 1$ is divisible by 3

Conclusion

Since $2^{2(1)} + 6(1) - 1$ is divisible by 3, and $2^{2(k+1)} + 6(k+1) - 1$ is divisible by 3 if $2^k + 6k - 1$ is divisible by 3, by the principle of mathematical induction the statement is true for all integers ≥ 1.

c) $P(x) = x^3 + 0x^2 - 2px + q$

i) sum of roots is 0

$\therefore$ $\alpha + \beta + (\alpha + \beta) = 0$
$2\alpha + 2\beta = 0$
$2\alpha = -2\beta$
$\alpha = -\beta$

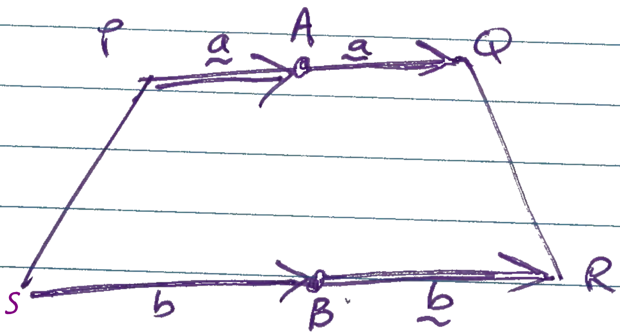
ii) $\alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) = -2p$

$$\alpha(-\alpha) + \alpha^2 + \alpha(-\alpha) + (-\alpha)(\alpha) + (-\alpha)(-\alpha) = -2p$$
$$-\alpha^2 + \alpha^2 - \alpha^2 - \alpha^2 + \alpha^2 = -2p$$
$$-\alpha^2 = -2p$$
$$\alpha^2 = 2p$$
$$\alpha = \pm\sqrt{2p}$$

$\therefore \alpha + \beta = \sqrt{2p} - \sqrt{2p} = 0$

The roots are $-\sqrt{2p}$, $\sqrt{2p}$ and 0.

d) i)



$$\vec{QR} = \vec{QA} + \vec{AB} + \vec{BR}$$
$$= -\underset{\sim}{a} + \vec{AB} + \underset{\sim}{b}$$
$$= \vec{AB} + \underset{\sim}{b} - \underset{\sim}{a}$$

ii) $\vec{PS} = \vec{PA} + \vec{AB} + \vec{BS}$

$$= \underset{\sim}{a} + \vec{AB} - \underset{\sim}{b}$$
$$= \vec{AB} + (\underset{\sim}{a} - \underset{\sim}{b})$$

so $\vec{AB} = \vec{PS} - (\underset{\sim}{a} - \underset{\sim}{b})$ ... (1)

and from i)

$\vec{AB} = \vec{QR} - (\underset{\sim}{b} - \underset{\sim}{a})$ ... (2)

(1) + (2)

$$2\vec{AB} = \vec{PS} + \vec{QR} - (\underset{\sim}{a} - \underset{\sim}{b}) - (\underset{\sim}{b} - \underset{\sim}{a})$$

$$2\vec{AB} = \vec{PS} + \vec{QR} \quad -\underset{\sim}{a} + \underset{\sim}{k} - \underset{\sim}{b} + \underset{\sim}{a}$$

$$2\vec{AB} = \vec{PS} + \vec{QR}$$

$$\vec{AB} = \frac{1}{2}(\vec{PS} + \vec{QR})$$

12/ i) $R\sin(x+\alpha) = \sqrt{3}\sin x + 1\cos x$

$R\sin x\cos\alpha + R\cos x\sin\alpha = \sqrt{3}\sin x + 1\cos x$

$(R\cos\alpha)\sin x + (R\sin\alpha)\cos x = \sqrt{3}\sin x + 1\cos x$

$\therefore\ R\sin\alpha = 1 \quad \ldots\ldots (1)$

$R\cos\alpha = \sqrt{3} \quad \ldots\ldots (2)$

$(1) \div (2) \quad \tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore\ \alpha = \pi/6 \quad$ since $0 < \alpha < \frac{\pi}{2}$

$(1)^2 + (2)^2 \quad R^2(\sin^2\alpha + \cos^2\alpha) = 1 + 3$

$R^2 = 4$

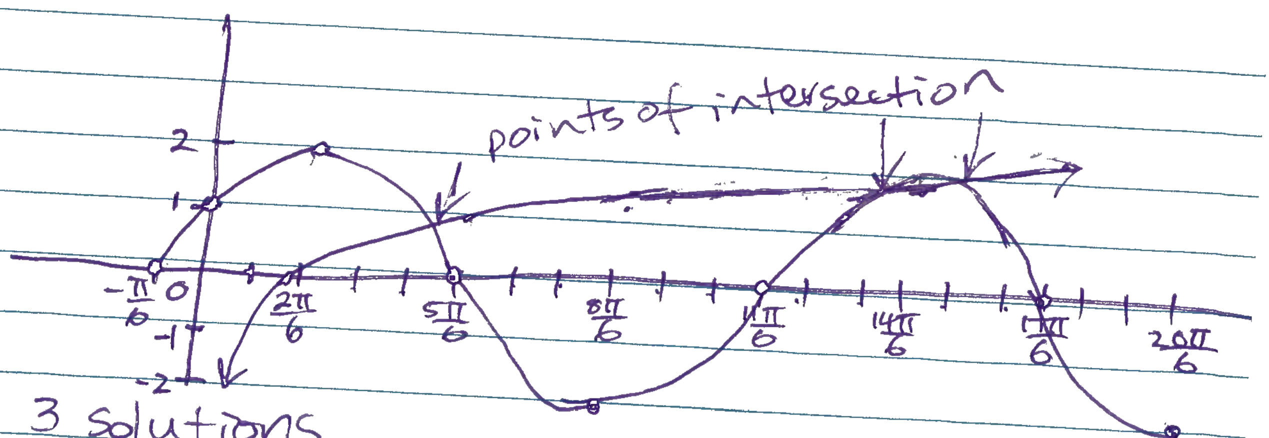
$R = \pm 2 \quad \therefore\ R = 2$ since $R > 0$

so $\sqrt{3}\sin x + \cos x = 2\sin(x + \frac{\pi}{6})$

ii) Missing numbers are

1   1.99   2.19

iii)



iv) 3 solutions

b) $(1+x)^m \Rightarrow {}^mC_r x^r$
$(1+x)^n \Rightarrow {}^nC_r x^r$

So to get x^4 we need on LHS

$${}^mC_0 x^0\, {}^nC_4 x^4 + {}^mC_1 x^1\, {}^nC_3 x^3 + {}^mC_2 x^2\, {}^nC_2 x^2$$
$$+ {}^mC_3 x^3\, {}^nC_1 x^1 + {}^mC_4 x^4\, {}^nC_0 x^0$$

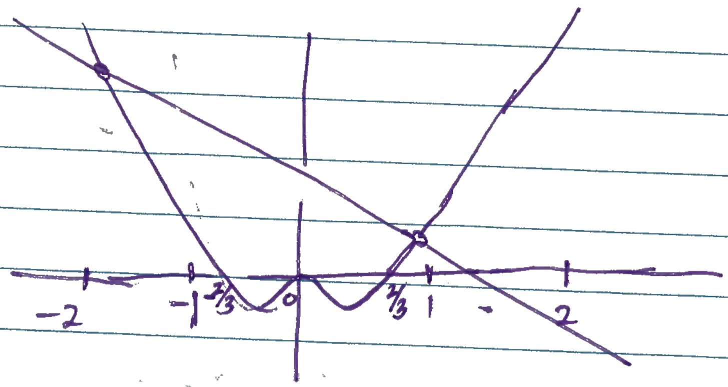
ie $\left({}^mC_0\,{}^nC_4 + {}^mC_1\,{}^nC_3 + {}^mC_2\,{}^nC_2 + {}^mC_3\,{}^nC_1 + {}^mC_4\,{}^nC_0\right) x^4$

On RHS

$${}^{m+n}C_4\, x^4$$

$$\therefore\ {}^{m+n}C_4 = {}^mC_0\,{}^nC_4 + {}^mC_1\,{}^nC_3 + {}^mC_2\,{}^nC_2 + {}^mC_3\,{}^nC_1 + {}^mC_4\,{}^nC_0$$
$$= {}^nC_4 + {}^mC_1\,{}^nC_3 + {}^mC_2\,{}^nC_2 + {}^mC_3\,{}^nC_1 + {}^mC_4$$
$$= {}^nC_4 + {}^nC_3\,{}^mC_1 + {}^nC_2\,{}^mC_2 + {}^nC_1\,{}^mC_3 + {}^mC_4$$

c) i)



for x-intercepts
$3x^3 - 2x^2 = 0$
$x^2(3x-2) = 0$
$x = 0, \frac{2}{3}$
$\therefore$ by reflection $-\frac{2}{3}$

ii) $y = 3(1)^3 - 2(1)^2 = 1 \Rightarrow$ intersect at $(1, 1)$
$y = 3|(-2)|^3 - 2|(-2)|^2 = 16 \Rightarrow$ intersect at $(-2, 16)$

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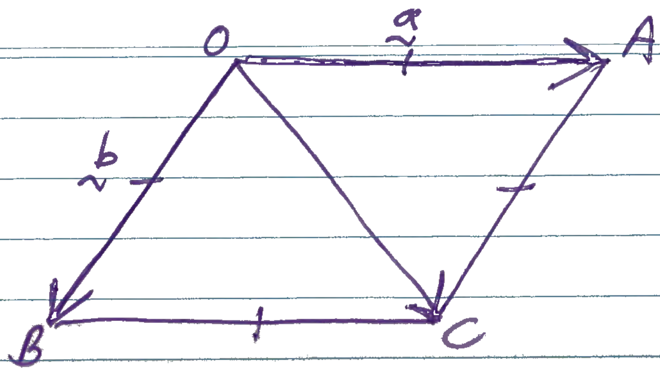
$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{16 - 1}{-2 - 1}(x - 1)$$
$$y - 1 = -5(x - 1)$$
$$y = -5x + 5 + 1$$
$$y = -5x + 6$$
$$\therefore m = -5, \quad b = 6$$

d) $6 \times 5 \times 4 = 120$ $\quad ({}^{6}P_{3})$

13/



$$\vec{OC} = \vec{OA} + \vec{AC}$$
$$= \vec{OA} + \vec{OB}$$
$$= \underset{\sim}{a} + \underset{\sim}{b}$$

$$\vec{OC} \cdot \vec{OA} = (\underset{\sim}{a} + \underset{\sim}{b}) \cdot \underset{\sim}{a}$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{a}||\underset{\sim}{b}| \cos \angle AOB$$

but $|\underset{\sim}{a}| = |\underset{\sim}{b}|$ $\therefore$ $\vec{OC} \cdot \vec{OA} = |\underset{\sim}{a}|^2 + |\underset{\sim}{a}|^2 \cos \angle AOB$

1

$$\vec{OC} \cdot \vec{OB} = (\underset{\sim}{a} \cdot \underset{\sim}{b}) \cdot \underset{\sim}{b}$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$$
$$= |\underset{\sim}{a}||\underset{\sim}{b}| \cos \angle AOB + |\underset{\sim}{b}|^2$$

but $|\underset{\sim}{a}| = |\underset{\sim}{b}|$ $\therefore$ $\vec{OC} \cdot \vec{OB} = |\underset{\sim}{a}|^2 + |\underset{\sim}{a}|^2 \cos \angle AOB$

2

From 1 & 2

$$\vec{OC} \cdot \vec{OA} = \vec{OC} \cdot \vec{OB}$$
$$|\vec{OC}||\underset{\sim}{a}| \cos \angle AOC = |\vec{OC}||\underset{\sim}{b}| \cos \angle BOC$$

but $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

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$$|\overrightarrow{OC}||\underset{\sim}{a}| \cos\angle AOC = |\overrightarrow{OC}||\underset{\sim}{a}| \cos\angle BOC$$

$$\therefore \cos\angle AOC = \cos\angle BOC$$

Now both $\angle AOC$ and $\angle BOC$ are in the range from 0 to $180°$.

$$\therefore \angle AOC = \angle BOC$$

b) $$V = \pi \int_0^h \left(\frac{3}{\sqrt{9+x^2}}\right)^2 dx$$

$$= \pi \int_0^h \frac{9}{9+x^2}\, dx$$

$$= 9\pi \int_0^h \frac{1}{9+x^2}\, dx$$

$$= 9\pi \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^h$$

$$= 3\pi \left[\tan^{-1}\frac{h}{3} - \tan^{-1}\frac{0}{3}\right]$$

$$= 3\pi \tan^{-1}\left(\frac{h}{3}\right) \text{ cubic units}$$

c) $\dfrac{dh}{dt} = 3$ cm/s find $\dfrac{dV}{dt}$ when $h=6$

$$V = 3\pi \tan^{-1}\left(\frac{h}{3}\right)$$

$$\frac{dV}{dh} = 3\pi \cdot \frac{\frac{1}{3}}{1+\left(\frac{h}{3}\right)^2}$$

$$\frac{dv}{dh} = 3\pi \frac{\frac{1}{3}}{\frac{9+h^2}{9}}$$

$$= \frac{9\pi}{9+h^2}$$

when $h=6$ $\quad \frac{dv}{dh} = \frac{9\pi}{9+6^2} = \frac{9\pi}{45} = \frac{\pi}{5}$

Now $\quad \frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$

$$= \frac{\pi}{5} \times 3$$

$$= 0.6\pi cm^3/s$$

c) $\quad S = \frac{2000}{1+199e^{-0.4t}}$

$$= 2000\left(1+199e^{-0.4t}\right)^{-1}$$

$$\frac{ds}{dt} = 2000 \times (-1) \times 199 \times (-0.4) e^{-0.4t} \times \left(1+199e^{-0.4t}\right)^{-2}$$

$$= \frac{(-2000 \times 199 \times -0.4)e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2}$$

$$= \frac{2000 \times \left(0.4 \times 199e^{-0.4t}\right)}{\left(1+199e^{-0.4t}\right) \times \left(1+199e^{-0.4t}\right)}$$

$$= S \times \frac{\frac{2}{5} \times 199e^{-0.4t}}{\left(1+199e^{-0.4t}\right)}$$

$$= \frac{S}{5}\left[\frac{2 \times 199e^{-0.4t}}{1+199e^{-0.4t}}\right]$$

$$= \frac{S}{5}\left[\frac{2\times1000}{1000} \times \frac{199e^{-0.4t}}{1+199e^{-0.4t}}\right]$$

$$= \frac{S}{5}\left[\frac{2000}{1+199e^{-0.4t}} \times \frac{199e^{-0.4t}}{1000}\right]$$

$$= \frac{S}{5}\left[S \times \frac{199e^{-0.4t}}{1000}\right]$$

$$= \frac{S}{5}\left[\frac{S}{1000} \times 199e^{-0.4t}\right]$$

Now, from $S = \frac{2000}{1+199e^{-0.4t}}$ we get

$$S(1+199e^{-0.4t}) = 2000$$

$$1+199e^{-0.4t} = \frac{2000}{S}$$

$$199e^{-0.4t} = \frac{2000}{S} - 1$$

$$= \frac{S}{5}\left[\frac{S}{1000} \times \left(\frac{2000}{S} - 1\right)\right]$$

$$= \frac{S}{5}\left[2 - \frac{S}{1000}\right]$$

ii) when $t=0$, $S = \frac{2000}{1+199} = 10$

as $t \to -\infty$ $S \to 0$

as $t \to \infty$ $S \to 2000$ $\therefore$ B

($t=0$, 2000 are given by horizontal slope line segments since they are constant solutions.)

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iii) When does $\frac{ds}{dt}$ reach a maximum?

$$\frac{ds}{dt} = \frac{S}{5}\left(2 - \frac{S}{1000}\right)$$
$$= \frac{2S}{5} - \frac{S^2}{5000}$$

$$= -\frac{S^2}{5000} + \frac{2}{5}S$$
$$= \left(-\frac{1}{5000}\right)S^2 + \frac{2}{5}S$$

This is a parabola (quadratic)
since $a<0$ it is concave down.

Max when $S = -\frac{b}{2a} = \frac{-\frac{2}{5}}{2\left(-\frac{1}{5000}\right)}$

$$= -\frac{2}{5} \div -\frac{2}{5000}$$
$$= -\frac{2}{5} \times \frac{5000}{-2}$$
$$= 1000$$

When $S = 1000$, $1000 = \frac{2000}{1+199e^{-0.4t}}$

$$1+199e^{-0.4t} = 2$$
$$199e^{-0.4t} = 1$$
$$e^{-0.4t} = \frac{1}{199}$$
$$e^{0.4t} = 199$$
$$0.4t = \ln 199$$
$$t = \frac{\ln 199}{0.4} = 13.233$$

About 13 days

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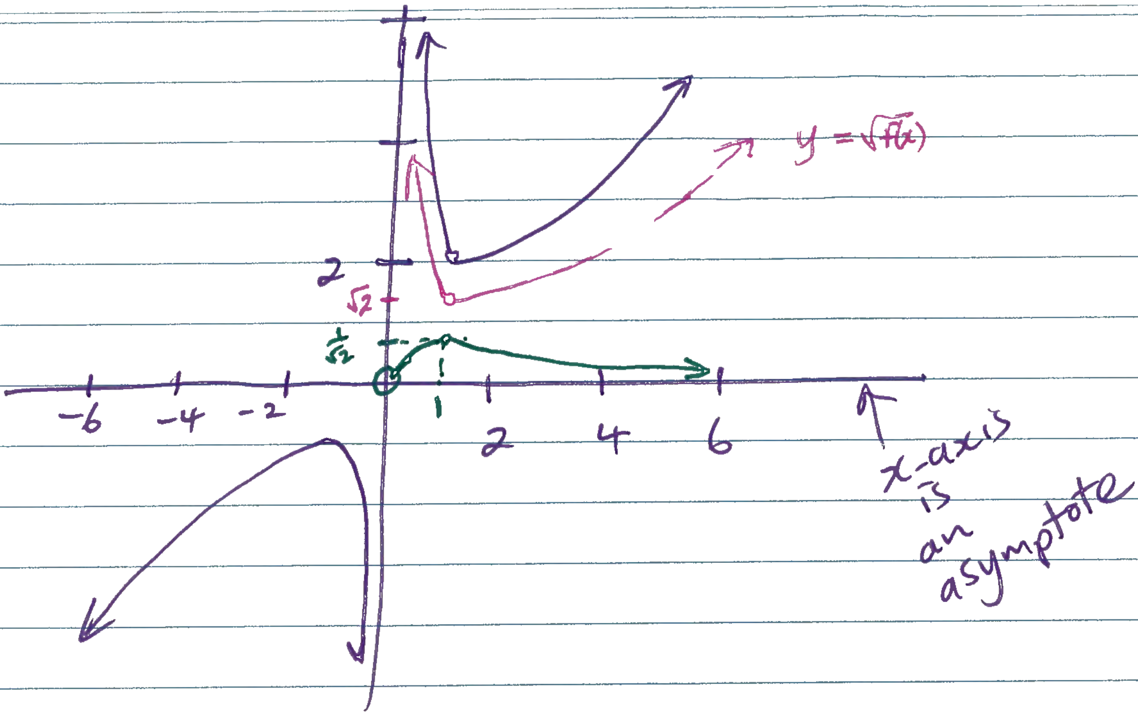
14/



$$f(x)=\frac{x^2+1}{x}$$

$$f'(x)=\frac{x.2x-(x^2+1).1}{x^2}=\frac{2x^2-x^2-1}{x^2}=\frac{x^2-1}{x^2}$$

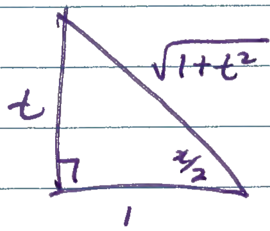
for T.P. $\frac{x^2-1}{x^2}=0$

$\therefore x=1$ or -1

Can't $\sqrt{}$ negative so we won't consider LHS of y-axis part of original curve.

$\therefore$ $x=1$ gives min T.P. on original & max T.P. on final curve.

b) i) let $t = \tan\frac{x}{2}$



$$\therefore \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}\left(\frac{\sqrt{1+t^2}}{1}\right)^2$$

$$= \frac{1+t^2}{2}$$

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1}{5+3\cos x}\,dx = \int \frac{1}{5+3\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\frac{5+5t^2+3-3t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{8+2t^2} \cdot 2dt$$

$$= \int \frac{1}{4+t^2} \cdot dt$$

$$= \frac{1}{2}\tan^{-1}\frac{t}{2} + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\left(\frac{x}{2}\right)\right) + c$$

ii) $\text{Area} = \int_0^{\pi} \frac{1}{5+3\cos x} \cdot dx$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)\right]_0^{\pi}$

$= \frac{1}{2}\left[\tan^{-1}\left(\frac{1}{2}\tan\frac{\pi}{2}\right) - \tan^{-1}\left(\frac{1}{2}\tan\frac{0}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$

$= \frac{\pi}{4}$

But $\tan\frac{\pi}{2}$ is actually undefined

So:

when $x = \pi$, $t = \tan\frac{\pi}{2}$

$A = \int_0^{\infty} \frac{1}{4+t^2} \, dt$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty}$

$= \frac{1}{2}\tan^{-1}(\infty) - \frac{1}{2}\tan^{-1} 0$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) - 0$

$= \frac{\pi}{4}$

c) i) $\underset{\sim}{a}(t) = 0\underset{\sim}{i} - 10\underset{\sim}{j}$

$$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\,.dt$$

$$\underset{\sim}{v}(t) = -10t\underset{\sim}{j} + c$$

when $t=0$, $\underset{\sim}{v}(t) = 90\cos 30°\underset{\sim}{i} + 90\sin 30°\underset{\sim}{j}$

$$= 90 \times \frac{\sqrt{3}}{2}\underset{\sim}{i} + 90 \times \frac{1}{2}\underset{\sim}{j}$$

$$= 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$$

so $\underset{\sim}{v}(0) = -10t\underset{\sim}{j} + c = 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$

$$-10(0)\underset{\sim}{j} + c = 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$$

$$c = 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = -10t\underset{\sim}{j} + 45\sqrt{3}\underset{\sim}{i} + 45\underset{\sim}{j}$$

$$\underset{\sim}{v}(t) = 45\sqrt{3}\underset{\sim}{i} + (45-10t)\underset{\sim}{j}$$

$$\underset{\sim}{s}(t) = \int \underset{\sim}{v}(t)\,dt$$

$$= \int \left[45\sqrt{3}\underset{\sim}{i} + (45-10t)\underset{\sim}{j}\right] dt$$

$$= 45\sqrt{3}t\underset{\sim}{i} + (45t - 5t^2)\underset{\sim}{j} + c$$

when $t=0$, $\underset{\sim}{s}(t) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

so $\underset{\sim}{s}(0) = 45\sqrt{3}(0)\underset{\sim}{i} + (45(0) - 5(0)^2)\underset{\sim}{j} + c = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$$c = 0$$

$\therefore$ $\underset{\sim}{s}(t) = 45\sqrt{3}t\underset{\sim}{i} + (45t - 5t^2)\underset{\sim}{j}$

ii) For the stone $\underset{\sim}{a}(t) = -10\underset{\sim}{j}$

$$\underset{\sim}{v}(t) = V\underset{\sim}{i} - 10t\underset{\sim}{j}$$

$$\underset{\sim}{s}(t) = Vt\underset{\sim}{i} - 5t^2\underset{\sim}{j} + 20\underset{\sim}{j}$$

$$\underset{\sim}{s}(t) = Vt\underset{\sim}{i} + (20-5t^2)\underset{\sim}{j}$$

Objects collide when position vectors are equal

BUT the stone wasn't fired until 5 seconds after the golf ball was hit.
∴ for the stone, relative to the golf-ball

$$\underset{\sim}{s}(t) = V(t-5)\underset{\sim}{i} + (20-5(t-5)^2)\underset{\sim}{j}$$

Now we can equate the position vectors of the golf-ball and the stone

$$V(t-5)\underset{\sim}{i} + (20-5(t-5)^2)\underset{\sim}{j} = (45\sqrt{3}\,t)\underset{\sim}{i} + (45t-5t^2)\underset{\sim}{j}$$

∴ $V(t-5) = 45\sqrt{3}t$ and $20-5(t-5)^2 = 45t-5t^2$
(equating components)

Using the $\underset{\sim}{j}$ components:

$$20-5(t-5)^2 = 45t-5t^2$$
$$20-5[t^2-10t+25] = 45t-5t^2$$
$$20-5t^2+50t-125 = 45t-5t^2$$

$50t - 105 = 45t$

$5t = 105$

$t = 21 \text{ s}$

iii) when $t = 21$ the $\underset{\sim}{i}$ components of the position vectors are equal

$\therefore \quad V(t-5) = 45\sqrt{3}\, t$

$V(21-5) = 45\sqrt{3}\,(21)$

$V = \dfrac{(45)\sqrt{3}(21)}{16}$

$V = \dfrac{945\sqrt{3}}{16} \text{ ms}^{-1}$

So for the stone

$v(t) = V\underset{\sim}{i} - 10t\,\underset{\sim}{j}$

$v(t) = \dfrac{945\sqrt{3}}{16}\underset{\sim}{i} - 10t\,\underset{\sim}{j}$

$v(16) = \dfrac{945\sqrt{3}}{16}\underset{\sim}{i} - 10(16)\,\underset{\sim}{j}$

$v(16) = \dfrac{945\sqrt{3}}{16}\underset{\sim}{i} - 160\,\underset{\sim}{j}$

The speed is $|V(21)| = \sqrt{\left(\dfrac{945\sqrt{3}}{16}\right)^2 + (160)^2}$

$= 189.9$

190 ms^{-1} to 3 significant figures.